

INVERSE SEMIGROUPS AND SHEU'S GROUPOID FOR THE ODD DIMENSIONAL QUANTUM SPHERES

S.SUNDAR

ABSTRACT. In this paper, we give a different proof of the fact that the odd dimensional quantum spheres are groupoid C^* algebras. We show that the C^* algebra $C(S_q^{2\ell+1})$ is generated by an inverse semigroup T of partial isometries. We show that the groupoid \mathcal{G}_{tight} associated to the inverse semigroup T in [1] is exactly the same as the groupoid considered in [6].

1. INTRODUCTION

Quantization of mathematical theories is a major theme of research today. The theory of Quantum groups and Noncommutative geometry are two prime examples in this program. The theory of compact quantum groups were initiated by Woronowicz in the late eighties in [8], [9], [10]. A main example in his theory is the quantum group $SU_q(n)$ and its homogeneous spaces. One of the problems in noncommutative geometry is to understand how these groups fit under Connes' formulation of NCG. Thus it becomes necessary to understand the C^* algebra of these quantum groups.

Vaksman and Soibelman studied the irreducible representations of C^* algebra of the quantum group $SU_q(n)$ in [7]. Exploiting their work, Sheu in [5] used the theory of groupoids and obtained certain composition sequences which is useful in understanding the structure of the C^* algebra of $SU_q(n)$. Then in [6] the question of whether C^* algebras of these quantum homogeneous spaces are in fact groupoid C^* algebras was raised. In [6], an affirmative answer is given for the quantum homogeneous space $S_q^{2n-1} := SU_q(n)/SU_q(n-1)$ called the odd dimensional quantum spheres. The purpose of this paper is to give an alternative proof of the same result. We use the theory of inverse semigroups developed in [1] to reconstruct the groupoid given in [6]. We believe that the proof is constructive as the groupoid in [6] is reconstructed from a combinatorial data naturally associated to S_q^{2n-1} .

Now we indicate the organisation of this paper. In the next section, we recall the basics of inverse semigroups and the groupoid associated to it without proofs. We refer to [1] for proofs. In section 3, we recall the definition of the C^* algebra of the odd

2010 *Mathematics Subject Classification.* **46L99, 20M18.**

Key words and phrases. Inverse semigroups, Groupoids, odd dimensional quantum spheres.

dimensional sphere $S_q^{2\ell+1}$ and associate a natural inverse semigroup to it. In sections 4-6, we work out the groupoid associated to the inverse semigroup and show that the groupoid is infact naturally isomorphic to Sheu's groupoid constructed in [6]. We end the paper by showing that the reduced C^* algebra associated to the groupoid is infact isomorphic to the C^* algebra of the odd dimensional quantum sphere $S_q^{2\ell+1}$.

2. INVERSE SEMIGROUPS AND THEIR GROUPOIDS

In this section, we briefly recall the construction of the groupoid associated to an inverse semigroup. We refer to [1] for proofs and details.

Definition 2.1. *An inverse semigroup T is an associative semigroup for every $s \in T$, there exists a unique element denoted s^* such that $s^*ss^* = s^*$ and $ss^*s = s$. Then $*$ is an involution which is antimultiplicative. We say that an inverse semigroup has 0 if there exists an element 0 such that $0.s = s.0 = 0$ for every $s \in T$.*

2.1. The unit space of the groupoid. Let T be an inverse semigroup with 0. We denote the set of projections in T by E i.e. $E := \{e \in T : e = e^* = e^2\}$. Then E is a commutative semigroup containing 0. Consider the set $\{0, 1\}$ as a multiplicative semigroup.

Definition 2.2. *Let T be an inverse semigroup with 0 and let E be its set of projections. A character of E is a nonzero map $x : E \rightarrow \{0, 1\}$ such that*

- (1) *The map x is a semigroup homomorphism, and*
- (2) *$x(0) = 0$.*

We denote the set of characters of E by \hat{E}_0 . The set of characters \hat{E}_0 is a locally compact Hausdorff topological space where the topology on \hat{E}_0 is the subspace topology inherited from the topology of $\{0, 1\}^E$.

The set of characters can alternatively be described in terms of filters by considering its support. For a character x , let $A_x := \{e \in E : x(e) = 1\}$. Then A_x is nonempty and has the following properties

- (1) $0 \notin A_x$,
- (2) If $e \in A_x$ and $f \geq e$ then $f \in A_x$, and
- (3) If $e, f \in A_x$ then $ef \in A_x$.

A nonempty subset A of E having properties (1), (2) and (3) is called a filter. Moreover, if A is a filter then the indicator function 1_A is a character. Thus there is a bijection between filters and characters. A filter is called an ultra filter if it is maximal. By Zorn's lemma ultra filters exist. Define

$$\hat{E}_\infty := \{x \in \hat{E}_0 : A_x \text{ is an ultrafilter} \}$$

and denote its closure by \hat{E}_{tight} .

2.2. The partial action of T on \hat{E}_0 . The inverse semigroup T acts naturally on \hat{E}_0 as partial homoemorphisms which we now explain. We let T act on \hat{E}_0 on the right as follows.

For $x \in \hat{E}_0$ and $s \in T$, define $(x.s)(e) = x(ses^*)$. Then

- (1) The map $x.s$ is a semigroup homomorphism, and
- (2) $(x.s)(0) = 0$.

But $x.s$ is nonzero if and only if $x(ss^*) = 1$. For $s \in T$, define the domain and range of s as

$$D_s := \{x \in \hat{E}_0 : x(ss^*) = 1\}$$

$$R_s := \{x \in \hat{E}_0 : x(s^*s) = 1\}$$

Note that both D_s and R_s are compact and open. Moreover s defines a homoemorphism from D_s to R_s with s^* as its inverse. Also observe that \hat{E}_{tight} is invariant under the action of T .

2.3. The groupoid \mathcal{G}_{tight} . Consider the transformation groupoid $\Sigma := \{(x, s) : x \in D_s\}$ with the composition and the inversion given by:

$$(x, s)(y, t) := (x, st) \text{ if } y = x.s$$

$$(x, s)^{-1} := (x.s, s^*)$$

Define an equivalence relation \sim on Σ as $(x, s) \sim (y, t)$ if $x = y$ and if there exists an $e \in E$ such that $x \in D_e$ for which $es = et$. Let $\mathcal{G} = \Sigma / \sim$. Then \mathcal{G} is a groupoid as the product and the inversion respects the equivalence relation \sim . Now we describe a topology on \mathcal{G} which makes \mathcal{G} into a topological groupoid.

For $s \in T$ and U an open subset of D_s , let $\theta(s, U) := \{[x, s] : x \in U\}$. We refer to [1] for the proof of the following two propositions. We denote $\theta(s, D_s)$ by θ_s . Then θ_s is homeomorphic to D_s and hence is compact, open and Hausdorff.

Proposition 2.3. *The collection $\{\theta(s, U) : s \in T, U \text{ open in } D_s\}$ forms a basis for a topology on \mathcal{G} . The groupoid \mathcal{G} with this topology is a topological groupoid whose unit space can be identified with \hat{E}_0 . Also one has the following.*

- (1) For $s, t \in T$, $\theta_s \theta_t = \theta_{st}$,
- (2) For $s \in T$, $\theta_s^{-1} = \theta_{s^*}$, and
- (3) The set $\{1_{\theta_s} : s \in T\}$ generates the C^* algebra $C^*(\mathcal{G})$.

We define the groupoid \mathcal{G}_{tight} to be the reduction of the groupoid \mathcal{G} to \hat{E}_{tight} .

3. THE ODD DIMENSIONAL QUANTUM SPHERES

Before we proceed let us fix some notations. Throughout we assume that $q \in (0, 1)$ and ℓ is a positive integer. We denote the set of non-negative integers by \mathbb{N} . We let \mathcal{H}_ℓ to denote the Hilbert space $\ell^2(\mathbb{N})^{\otimes \ell} \otimes \ell^2(\mathbb{Z})$. We denote the left shift operator on $\ell^2(\mathbb{N})$ by S and the right shift on $\ell^2(\mathbb{Z})$ by t . The number operator on $\ell^2(\mathbb{N})$ is denoted by N .

In this section we recall a few well known facts about the C^* algebra of the odd dimensional quantum spheres. The C^* -algebra $C(S_q^{2\ell+1})$ of the quantum sphere $S_q^{2\ell+1}$ is the universal C^* -algebra generated by elements $z_1, z_2, \dots, z_{\ell+1}$ satisfying the following relations (see [2]):

$$\begin{aligned} z_i z_j &= q z_j z_i, & 1 \leq j < i \leq \ell + 1, \\ z_i^* z_j &= q z_j z_i^*, & 1 \leq i \neq j \leq \ell + 1, \\ z_i z_i^* - z_i^* z_i + (1 - q^2) \sum_{k>i} z_k z_k^* &= 0, & 1 \leq i \leq \ell + 1, \\ \sum_{i=1}^{\ell+1} z_i z_i^* &= 1. \end{aligned}$$

Note that for $\ell = 0$, the C^* -algebra $C(S_q^{2\ell+1})$ is the algebra of continuous functions $C(\mathbb{T})$ on the torus and for $\ell = 1$, it is $C(SU_q(2))$.

Let $Y_{k,q}$ be the following operators on \mathcal{H}_ℓ :

$$(3.1) \quad Y_{k,q} = \begin{cases} \underbrace{q^N \otimes \dots \otimes q^N}_{k-1 \text{ copies}} \otimes \sqrt{1 - q^{2N}} S^* \otimes \underbrace{I \otimes \dots \otimes I}_{\ell+1-k \text{ copies}}, & \text{if } 1 \leq k \leq \ell, \\ \underbrace{q^N \otimes \dots \otimes q^N}_{\ell \text{ copies}} \otimes t, & \text{if } k = \ell + 1. \end{cases}$$

Then $\pi_\ell : z_k \mapsto Y_{k,q}$ gives a faithful representation of $C(S_q^{2\ell+1})$ on \mathcal{H}_ℓ for $q \in (0, 1)$ (see lemma 4.1 and remark 4.5, [2]). We let $Y_{k,0}$ to denote the limit of the operators $Y_{k,q}$ as q tends to zero. The formula for $Y_{k,0}$ is again the same as that of $Y_{k,q}$ where q^N stands for the rank one projection $p = |e_0\rangle\langle e_0|$.

Consider the unitary operator U on \mathcal{H} defined by $U(e_{m,z}) = e(m, z + \sum_{i=1} m_i)$. Define $Z_{k,q} := UY_{k,q}U^*$ for $q \in [0, 1]$. The representation $z_k \rightarrow Z_{k,q}$ of $C(S_q^{2\ell+1})$ is the one considered in [5] and in [6]. Let $A_\ell(q)$ be the image of $C(S_q^{2\ell+1})$ under this representation i.e. $A_\ell(q)$ is the C^* algebra generated by $Z_{k,q}$. We refer to [4] for the proof of the following proposition.

Proposition 3.1. *For any $q \in (0, 1)$, one has $A_\ell(0) = A_\ell(q)$.*

From now on, we simply denote $Z_{k,0}$ by Z_k . Note that Z_k 's are in fact partial isometries. Let us introduce more notations. For $m, n \in \mathbb{N}^\ell$ and $r \in \mathbb{Z}$, Let $B_k(r, m, n)$ be defined as follows:

$$(3.2) \quad B_k(r, m, n) = \begin{cases} \underbrace{S^{*m_1} p S^{n_1} \otimes S^{*m_2} p S^{n_2} \otimes \dots \otimes S^{*m_{k-1}} p S^{n_{k-1}}}_{k-1 \text{ copies}} \otimes S^{*m_k} S^{n_k} \otimes 1 \otimes t^{(\sum_{i=1}^k (m_i - n_i))} & \text{if } 1 \leq k \leq \ell \\ \underbrace{S^{*m_1} p S^{n_1} \otimes S^{*m_2} p S^{n_2} \otimes \dots \otimes S^{*m_\ell} p S^{n_\ell}}_{\ell \text{ copies}} \otimes t^{r + \sum_{i=1}^\ell (m_i - n_i)}, & \text{if } k = \ell + 1. \end{cases}$$

Note the following commutation relations.

If $i < j$ then

$$B_i(r, m, n) B_j(r', m', n') = \delta_{(n_1, n_2, \dots, n_{i-1}), (m'_1, m'_2, \dots, m'_{i-1})} 1_{[n_i, \infty)}(m'_i) B_k(r', m'', n')$$

where $m'' = (m_1, m_2, \dots, m_{i-1}, m_i + m'_i - n_i, m'_{i+1}, \dots, m'_\ell)$.

If $i \leq \ell$ and $n_i \leq m'_i$ then

$$B_i(r, m, n) B_i(r', m', n') = \delta_{(n_1, n_2, \dots, n_{i-1}), (m'_1, m'_2, \dots, m'_{i-1})} B_i(r', m'', n')$$

where $m'' := (m_1, m_2, \dots, m_{i-1}, m_i + m'_i - n_i, m'_{i+1}, \dots, m'_\ell)$.

If $i \leq \ell$ and $n_i < m'_i$ then

$$B_i(r, m, n) B_i(r', m', n') = \delta_{(n_1, n_2, \dots, n_{i-1}), (m'_1, m'_2, \dots, m'_{i-1})} B_i(r', m, n'')$$

where $n'' := (n'_1, n'_2, \dots, n'_{i-1}, n'_i + n_i - m'_i, n'_{i+1}, \dots, n'_\ell)$.

Finally, $B_{\ell+1}(r, m, n) B_{\ell+1}(r', m', n') = \delta_{n, m'} B_{\ell+1}(r + r', m, n')$. Also $B_i(r, m, n)^* := B_i(-r, n, m)$. It is clear from the above commutation relations that the set $T := \{0\} \cup \{B_i(r, m, n) : 1 \leq i \leq \ell + 1, r \in \mathbb{Z}, m, n \in \mathbb{N}^k\}$ is an inverse semigroup of partial isometries.

Proposition 3.2. *The set $T := \{0\} \cup \{B_i(r, m, n) : 1 \leq i \leq \ell + 1, r \in \mathbb{Z}, m, n \in \mathbb{N}^k\}$ is an inverse semigroup of partial isometries. Moreover T is generated by $\{Z_i : 1 \leq i \leq \ell + 1\}$.*

Proof. As already observed T is an inverse semigroup of partial isometries. Let e_i be the ℓ tuple which is 1 on the i th coordinate and zero elsewhere. Then $Z_k := B(0, e_k, 0)$ for $k \leq \ell$ and $Z_{\ell+1} = B_{\ell+1}(1, 0, 0)$. Thus Z_k 's are in T . Moreover,

$$0 = Z_1^* Z_2$$

$$B_i(r, m, n) = Z_1^{m_1} Z_2^{m_2} \dots Z_i^{m_i} Z_i^{*n_i} Z_{i-1}^{*n_{i-1}} \dots Z_1^{*n_1} \text{ if } i \leq \ell$$

$$B_{\ell+1}(r, m, n) = Z_1^{m_1} Z_2^{m_2} \dots Z_\ell^{m_\ell} (Z_{\ell+1}^{r_+} Z_{\ell+1}^{r_-}) Z_\ell^{*n_\ell} Z_{\ell-1}^{*n_{\ell-1}} \dots Z_1^{*n_1}.$$

where r_+ and r_- denote the positive and negative parts of r . Thus every element in T is a word in Z_i 's. This completes the proof. \square

4. THE TIGHT CHARACTERS FOR THE INVERSE SEMIGROUP T

In this section, we describe the tight characters of the inverse semigroup T defined in Proposition 3.2. The set of projections of T is denoted by E and the set of characters of E by \hat{E}_0 . Consider the one point compactification $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ of \mathbb{N} . We denote the projection from $\overline{\mathbb{N}}^\ell$ onto the first r components by π_r .

Let $p_i(m) := B_i(0, m, m)$. Then $E := \{0\} \cup \{p_i(m) : 1 \leq i \leq \ell + 1, m \in \mathbb{N}^k\}$. First observe that if Λ is a subsemigroup of E not containing 0 then the set

$$A_\Lambda := \{f \in E : f \geq e \text{ for some } e \in \Lambda\}$$

is a filter.

Let $k \in \overline{\mathbb{N}}^\ell$ be given. Let r be the least positive integer for which $k_{r+1} = \infty$. (For an ℓ tuple k , we set $k_{\ell+1} = \infty$). Define $\Lambda_k = \{p_{r+1}(\pi_r(k), n) : n \in \mathbb{N}^\ell\}$. It is easy to see that Λ_k is a subsemigroup of E not containing 0. Then A_{Λ_k} is a filter and thus gives rise to a character. We denote the character associated to A_{Λ_k} by $\phi(k)$. The following lemma gives a closed formula for $\phi(k)$.

Lemma 4.1. *Let $k \in \overline{\mathbb{N}}^\ell$ be given. The character $\phi(k)$ is given by*

$$\phi(k)(p_i(m)) := \begin{cases} \delta_{\pi_{i-1}(m), \pi_{i-1}(k)} 1_{[0, k_i]}(m_i) & \text{if } i \leq \ell \\ \delta_{m, k} & \text{if } i = \ell + 1 \end{cases}$$

Proof. Let r be the least positive integer for which $k_{r+1} = \infty$. Observe $p_i(m) \geq p_{r+1}(\pi_r(k), n)$ for some n if and only if $i \leq r + 1$, $\pi_{i-1}(m) = \pi_{i-1}(k)$ and $m_i \leq k_i$. Now the proof follows. \square

An immediate consequence of the above lemma is that the map $\phi : \overline{\mathbb{N}}^\ell \rightarrow \hat{E}_0$ is continuous. In the next proposition we show that the image of ϕ is exactly \hat{E}_∞ .

Proposition 4.2. *The image of the map ϕ is infact \hat{E}_∞ .*

Proof. First we show that the image of ϕ is contained in \hat{E}_∞ . Let $k \in \overline{\mathbb{N}}^\ell$ be given and let r be the least nonnegative integer for which $k_{r+1} = \infty$. Recall that

$$\Lambda_k := \{p_{r+1}(\pi_r(k), n) : n \in \mathbb{N}^{\ell-r}\}$$

We denote $\phi(k)$ by x . We claim that A_x is an ultrafilter. Suppose that there exists a character say y such that $A_x \subset A_y$. Then we need to show that $x = y$ or $A_x = A_y$. Since $x = 1$ on Λ_k and $A_x \subset A_y$, it follows that $y = 1$ on Λ_k . If $\pi_r(m) \neq \pi_r(k)$ then $p_{r+1}(\pi_r(m), v)$ is orthogonal to Λ_k . Hence x and y vanishes on $p_{r+1}(\pi_r(m), v)$. Thus $x(p_{r+1}(m)) = y(p_{r+1}(m))$ for every $m \in \mathbb{N}^\ell$.

Now let $i > r + 1$ be given. Let $m \in \mathbb{N}^\ell$. Choose an ℓ tuple n such that $\pi_r(n)y = \pi_r(k)$ and $n_{r+1} > m_{r+1}$. Then $p_i(m)$ and $p_{r+1}(n)$ are orthogonal. But $x = y = 1$ on $p_{r+1}(n)$. Thus x and y vanishes on $p_i(m)$.

Now let $i \leq r$ and $m \in \mathbb{N}^\ell$. If $m_i > k_i$ then $p_i(m)$ is orthogonal to Λ_k . But $x = y = 1$ on Λ_k . Thus x and y vanishes on $p_i(m)$ if $m_i > k_i$. Now suppose $m_i \leq k_i$. If $\pi_{i-1}(m) \neq \pi_{i-1}(k)$ then $p_i(m)$ is again orthogonal to Λ_k and thus x and y vanishes on $p_i(m)$. Consider now the case where $m_i \leq k_i$ and $\pi_{i-1}(m) = \pi_{i-1}(k)$. Then $x(p_i(m)) = 1$ by definition and since $A_x \subset A_y$, it follows that $y(p_i(m)) = 1$. Thus we have shown that $x(p_i(m)) = y(p_i(m))$ for every i and m . Hence $x = y$ or in otherwords A_x is an ultrafilter. This proves that $\phi(\overline{\mathbb{N}}^\ell)$ is contained in \hat{E}_∞ .

Now let us prove that \hat{E}_∞ is contained in the range of ϕ . Let $x \in \hat{E}_\infty$ be given. Let r be the largest nonnegative integer for which there exists a k' such that $x = 1$ on $p_{r+1}(k')$. Choose k such that $\pi_r(k) = \pi_r(k')$ and $k_{r+1} = \infty$. We claim that $A_x \subset A_{\phi(k)}$. Then the maximality of A_x forces the equality $x = \phi(k)$.

Let $i \leq r + 1$ be given. Consider an ℓ tuple m such that either $m_i > k_i$ or $\pi_{i-1}(m) \neq \pi_{i-1}(k)$. Then $p_i(m)$ is orthogonal to $p_{r+1}(k')$. Hence $x(p_i(m)) = 0$ if either $m_i > k_i$ or $\pi_{i-1}(m) \neq \pi_{i-1}(k)$. Also x vanishes on $p_i(m)$ if $i > r + 1$ by the choice of r . Thus we have shown $A_{\phi(k)}^c \subset A_x^c$. Hence A_x is contained in $A_{\phi(k)}$. Since A_x is maximal, it follows that $x = \phi(k)$. This completes the proof. \square

Corollary 4.3. *The set \hat{E}_∞ is compact and $\hat{E}_{tight} = \hat{E}_\infty$.*

Proof. The proof follows from the fact that ϕ is continuous, $\overline{\mathbb{N}}^\ell$ is compact and from Proposition 4.2. \square .

Now define an equivalence relation on $\overline{\mathbb{N}}^\ell$ as follows:

$$k \sim k' \text{ if there exists } r \geq 0 \text{ such that } \pi_r(k) = \pi_r(k') \text{ and } k_{r+1} = k'_{r+1}.$$

We show that \hat{E}_{tight} is homeomorphic to the quotient space $\overline{\mathbb{N}}^\ell / \sim$ in the next proposition.

Proposition 4.4. *The map $\phi : \overline{\mathbb{N}}^\ell \rightarrow \hat{E}_{tight}$ factors through the quotient $\overline{\mathbb{N}}^\ell / \sim$ to give a map $\tilde{\phi} : \overline{\mathbb{N}}^\ell / \sim \rightarrow \hat{E}_{tight}$. Also the maps $\tilde{\phi}$ is a homeomorphism.*

Proof. It is clear from the definition and from Lemma 4.1 that ϕ factors through the quotient to give a map $\tilde{\phi}$. Since ϕ is continuous, it follows that $\tilde{\phi}$ is continuous. We now show that $\tilde{\phi}$ is one-one.

Let k, k' be such that $\phi(k) = \phi(k')$. Let r_k (resp. $r_{k'}$) be the least nonnegative integer for which $k_{r_k+1} = \infty$ (resp. $k'_{r_{k'}+1} = \infty$). Then r_k is the last integer for which there exists an m such that $\phi(k)$ is 1 on $p_{r_k+1}(m)$. Thus $r_k = r_{k'}$. Moreover $\phi(k)$ is 1 on $p_{r_k+1}(\pi_{r_k}(k), u)$. Thus $\phi_{k'}$ is 1 on $p_{r_k+1}(\pi_{r_k}(k), u)$. Now Lemma 4.1 implies that $\pi_{r_k}(k) = \pi_{r_k}(k')$. Hence $k \sim k'$. This proves that $\tilde{\phi}$ is one-one.

Now Proposition 4.2 implies that $\tilde{\phi}$ is onto. As $\overline{\mathbb{N}}^\ell / \sim$ is compact and \hat{E}_{tight} is Hausdorff, it follows that $\tilde{\phi}$ is infact a homeomorphism. This completes the proof. \square

5. SHEU'S GROUPOID

In this section, we recall the groupoid for the odd dimensional quantum spheres $S_q^{2\ell+1}$ described in [6]. Consider the transformation groupoid $\mathbb{Z} \times (\mathbb{Z}^\ell \times \overline{\mathbb{Z}}^\ell)$ where \mathbb{Z}^ℓ acts on $\overline{\mathbb{Z}}^\ell$ by translation. Let \mathcal{F} be the restriction of the transformation groupoid to $\overline{\mathbb{N}}^\ell$. Define

$$\Sigma := \{(z, x, w) \in \mathcal{F} : w_i = \infty \Rightarrow x_{i+1} = \cdots = x_\ell = 0 \text{ and } z = -\sum_{j=1}^i x_j\}$$

Then Σ is an open subgroupoid of \mathcal{F} . Define an equivalence relation \sim on Σ as follows:

$$(z, x, w_1, w_2, \cdots, w_{i-1}, \infty, \cdots) \sim (z, x, w_1, w_2, \cdots, w_{i-1}, \infty, \infty, \cdots, \infty).$$

Let $\mathcal{G} := \Sigma / \sim$. Then the multiplication and the inversion on Σ factors through the equivalence relation making \mathcal{G} into a groupoid. When \mathcal{G} is given the quotient topology, it becomes a topological groupoid which is the groupoid described in [6].

6. THE GROUPOID \mathcal{G}_{tight} OF THE INVERSE SEMIGROUP T

In this section, we show that the groupoid \mathcal{G}_{tight} of the inverse semigroup T is isomorphic with Sheu's groupoid described in the previous section. For an ℓ tuple m , we set $m_{\ell+1} = \infty$. We define a map $\psi : \Sigma \rightarrow \mathcal{G}_{tight}$ as follows:

Let $(z, x, w) \in \Sigma$ be given. Let r be the least nonnegative integer for which $w_{r+1} = \infty$. Then ψ on (z, x, w) is given by

$$\psi(z, x, w) := [(\phi(w), B_{r+1}(t, m, n))]$$

where t, m, n are given by $t := z + \sum_{j=1}^r x_j$, $m := (w_1, w_2, \cdots, w_r, |x_{r+1}|, 0, \cdots, 0)$ and $n := (x_1 + w_1, x_2 + w_2, \cdots, x_r + w_r, x_{r+1} + |x_{r+1}|, 0, \cdots, 0)$. Observe that ψ is well defined as $\pi_r(w) = \pi_r(m)$ and $w_{r+1} = \infty$.

Let us introduce the following notation. For $m, n \in \mathbb{N}^\ell$ and $0 \leq r \leq \ell$, let $A_{r+1}(m, n) := S^{*m_1} p S^{n_1} \otimes \cdots \otimes S^{*m_r} p S^{n_r} \otimes S^{*m_{r+1}} S^{n_{r+1}} \otimes 1$. We consider $A_{r+1}(m, n)$ as an operator on $\ell^2(\mathbb{N}^\ell)$.

Proposition 6.1. *The map ψ is continuous and ψ factors through the equivalence relation \sim . Let $\tilde{\psi}$ be the induced map from $\mathcal{G} \rightarrow \mathcal{G}_{tight}$. Then $\tilde{\psi}$ is infact a groupoid isomorphism.*

Proof. First we show that ψ factors through the equivalence relation. Let $(z, x, w) \sim (z, x, w')$ and let r (resp. r') be the least nonnegative integer for which $w_{r+1} = \infty$ (resp

$w'_{r'+1} = \infty$). By definition $r = r'$ and $\pi_r(w) = \pi_r(w')$. Then by Proposition 4.4, it follows that $\phi(w) = \phi(w')$. Since the definition of ψ involves only the first r components of w , $\psi(z, x, w) = \psi(z, x, w')$. This proves that $\tilde{\psi}$ is well defined.

The map $\tilde{\psi}$ is one-one. Suppose that $\psi(z, x, w) = \psi(z', x', w')$. Again let r and r' be the least nonnegative integer for which w_{r+1} and $w'_{r'+1}$ are both ∞ . Then r is the largest integer for which there exists an m such that $\phi(w)$ is 1 on $p_{r+1}(m)$. Since $\phi(w) = \phi(w')$, we have $r = r'$ and $\pi_r(w) = \pi_r(w')$. As $\psi(z, x, w) = \psi(z', x', w')$, it follows that there exists a projection e such that $\phi(w)(e) = 1$ and $e(t^z \otimes A_{r+1}(m, n)) = e(t^{z'} \otimes A_{r+1}(m', n'))$. But $\phi(w)(e) = 1$ implies that $e \geq p_{r+1}(\pi_r(w), 0)$. Hence we can choose e to be $p_{r+1}(\pi_r(w), 0)$. Thus it follows that $z = z'$ and $A_{r+1}(m, n) = A_{r+1}(m', n')$. Thus $m = m'$ and $n = n'$ which in turn implies $x_i = x'_i$ for $i \leq r+1$. Since $(z, x, w) \in \Sigma$, it follows that $x_i = x'_i = 0$ for $i \geq r+2$. Thus we have shown that $(z, x, w) \sim (z', x', w')$. Hence $\tilde{\psi}$ is one-one.

The map $\tilde{\psi}$ is onto. First note that if $a - b = c - d$ then there exists a projection $e = S^{*(b+c)}S^{b+c}$ such that $eS^{*b}S^a = eS^{*d}S^c$. Hence in the definition of ψ we can change the $r+1$ th components of m and n such that $n_{r+1} - m_{r+1} = x_{r+1}$. Let $[(\phi(w), B_i(s, m, n))]$ be an element in \mathcal{G}_{tight} . Let r be the first nonnegative integer for which $w_{r+1} = \infty$. Then $i \leq r+1$. By premultiplying by $p_{r+1}(\pi_r(w), 0)$ we can assume that $i = r+1$ and m is such that $\pi_r(w) = \pi_r(m)$. Now if $r \leq \ell - 1$ then, for $z = \sum_j (m_j - n_j)$, x such that $\pi_{r+1}(x) = \pi_{r+1}(n) - \pi_{r+1}(m)$ and $x_i = 0$ for $i \geq r+2$, $\psi(z, x, w) = [(\phi(w), B_{r+1}(s, m, n))]$. If $r = \ell$ then with $z = s + \sum_{j=1}^{\ell} (m_j - n_j)$ and $x_j = n_j - m_j$ one has $\psi(z, x, w) = [(\phi(w), B_{\ell+1}(s, m, n))]$. This proves that $\tilde{\psi}$ is onto.

The map $\tilde{\psi}$ is continuous. Let (z^n, x^n, w^n) be a sequence in Σ converging to $(z, x, w) \in \Sigma$. Let r be the least nonnegative integer for which $w_{r+1} = \infty$. Then eventually $(z^n, x^n, \pi_r(w^n))$ coincides with $(z, x, \pi_r(w))$. Suppose that $\theta(s, U)$ is an open set containing $\psi(z, x, w)$. Without loss of generality we can assume that $s := t^z \otimes A_{r+1}(m, n)$ where m, n are defined as in the definition of ψ . Since U is an open set containing $\phi(w)$ and as ϕ is continuous, it follows that $\phi(w^n) \in U$ eventually. Let r_n be the least nonnegative integer for which $w^n_{r_n+1} = \infty$. Then $r_n \geq r$. Let m^n and n^n be as in the definition of ψ for (z, x, w^n) . If $e_n := p_{r_n+1}(m^n)$, then $\phi(w^n) \in D_{e_n}$ and $e_n(t^z \otimes A_{r_n+1}(m^n, n^n)) = e_n(t^z \otimes A_{r+1}(m, n))$ eventually. Thus eventually $\psi(z^n, x^n, w^n) = [(\phi(w^n), s)] \in \theta(s, U)$. This proves that $\tilde{\psi}$ is continuous.

We leave it to the reader to check that $\tilde{\psi}$ is a homeomorphism and it is infact a groupoid homomorphism. This completes the proof. \square

7. ISOMORPHISM BETWEEN $C(S_q^{2\ell+1})$ AND $C_{red}^*(\mathcal{G})$

We complete the discussion by showing that $C(S_q^{2\ell+1})$ is isomorphic to the reduced C^* algebra of the groupoid \mathcal{G} where \mathcal{G} is the Sheu's groupoid considered in Section 5. We also identify \mathcal{G} with \mathcal{G}_{tight} . Let r denote the range map and let $\mathcal{G}^0 := r^{-1}(\phi(0))$. Then $\mathcal{G}^0 := \{(z, x, 0) : x_i \geq 0\}$. Thus $L^2(\mathcal{G}^0)$ is naturally isomorphic to $\ell^2(\mathbb{N}^\ell) \otimes \ell^2(\mathbb{Z})$. Consider the representation π of $C_{red}^*(\mathcal{G})$ on $L^2(\mathcal{G}^0)$ defined by:

For $f \in C_c(\mathcal{G})$ and $\xi \in L^2(\mathcal{G}^0)$,

$$(\pi(f)\xi)(\gamma) = \sum_{\gamma_1 \in \mathcal{G}^0} f(\gamma^{-1}\gamma_1)\xi(\gamma_1)$$

The representation π is equivalent to $Ind(\delta_0)$ given in [3] and it is faithful since the largest open set which is also invariant for which $\delta_0(U) = 0$ is the empty set. (Refer to [3] for more details.)

Let $\Sigma_{fin} \subset \Sigma$ be the finite part i.e. $\Sigma_{fin} := \{(z, x, w) : w_i < \infty\}$. Then Σ_{fin} is a subset of \mathcal{G} . For $1 \leq k \leq \ell + 1$, denote the set $\theta_{Z_k^*}$ by θ_k . Now it is easily verifiable that $\theta_k \cap \Sigma_{fin} := \{(-1, e_k, (0, 0, \dots, 0, w_k, \dots, w_\ell))\}$ where e_k is the ℓ tuple which is 1 at the k th place and zero elsewhere. From this observation it is easy to show that $\pi(1_{\theta_k}) = Z_k^*$. Since 1_{θ_k} generate the C^* algebra $C_{red}^*(\mathcal{G}_{tight})$, it follows that $C(S_q^{2\ell+1})$ is isomorphic to $C_{red}^*(\mathcal{G})$.

REFERENCES

- [1] Ruy Exel. Inverse semigroups and combinatorial C^* algebras. *Bull.Braz.Math.Soc.(NS)*, 39(2):191–313, 2008.
- [2] Jeong Hee Hong and Wojciech Szymanski. Quantum spheres and projective spaces as graph algebra. *Commun.Math.Phys.*, 232:157–188, 2002.
- [3] Paul S. Muhly and Jean N. Renault. C^* -algebras of multivariable Wiener-Hopf operators. *Trans.Amer.Math.Soc.*, 274(1):1–44, Nov 1982.
- [4] ArupKumar Pal and S.Sundar. Regularity and dimension spectrum of the equivariant spectral triple for the odd dimensional quantum spheres. *Journal of Noncommutative geometry(to appear)*.
- [5] Albert J.L. Sheu. Compact quantum groups and groupoid C^* algebras. *Journal of Functional Analysis*, 144:371–393, 1997.
- [6] Albert J.L. Sheu. Quantum spheres as groupoid C^* algebras. *Quarterly J.Math*, 48:503–510, 1997.
- [7] L. L. Vaksman and Ya. S. Soibel'man. Algebra of functions on the quantum group $SU(n + 1)$, and odd-dimensional quantum spheres. *Algebra i Analiz*, 2(5):101–120, 1990.
- [8] S. L. Woronowicz. Compact matrix pseudogroups. *Comm. Math. Phys.*, 111(4):613–665, 1987.
- [9] S. L. Woronowicz. Tannaka-Kreĭn duality for compact matrix pseudogroups. Twisted $SU(N)$ groups. *Invent. Math.*, 93(1):35–76, 1988.
- [10] S. L. Woronowicz. Compact quantum groups. In *Symétries quantiques (Les Houches, 1995)*, pages 845–884. North-Holland, Amsterdam, 1998.

Acknowledgement: This work was undertaken when I was a postdoc at Universite de Caen, France from Sep 2010- Jan 2011. I thank Prof. Emmanuel Germain for his support and encouragement during my stay at Caen. I would also like to acknowledge CNRS for the financial support.

EMAIL: sundarsobers@gmail.com

Visiting Institute of Mathematical Sciences, Chennai.